# On magic graphs and magic stars

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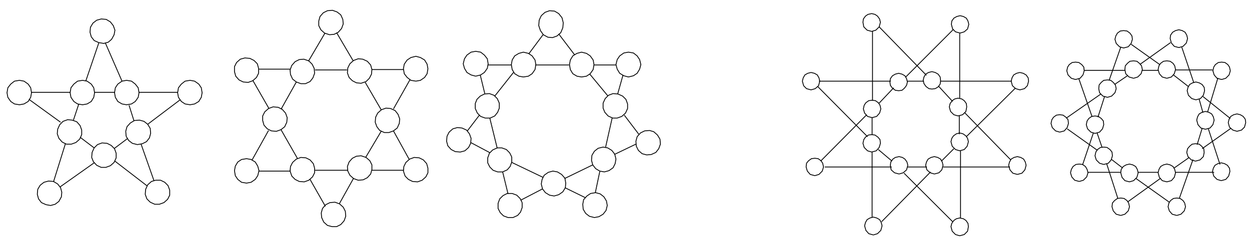
**Abstract:** In 1962, Jiří Sedláček discovered certain connections between magic squares and a special class of edge-labeled graphs, and called them *magic graphs*. Since then, many mathematicians have studied magic graphs defined in different ways. In this contribution, we will apply some results on magic graphs at brain-twisters called magic stars.

You have probably already met with the following brain-twisters whose author is the English mathematician Henry Dudeney [2].

Problem 1: Into the circlets of the star $S_5$ (Figure 1) write ten different numbers from the set {1, 2, 3, …., 12} in such a way that the sum of the four numbers on each line is twenty-four.

Problem 2: Into the circlets of the star $S_8$ write numbers 1, 2,…., 16 so that the sum of the four numbers on each line is thirty.



$\mathbb{S}_5$ $\mathbb{S}_6$ $\mathbb{S}_7$ $\mathbb{T}_8$ $\mathbb{T}_{10}$

Figure 1

Both problems are solved and a generalization is described in [6]. In this paper we decsribe some connections between magic graphs and stars.

In Figure 1 five *n*-sided (*n*-vertices) stars of two types are depicted. The first three stars are denoted $S_n$, $n=5,6,7$ and the two others $T_n$, $n=8,10$. Each star contains $2n$ circlets in such a way that there are four on each line. Both types of stars arise from a regular *n*-gon, in the vertices of which there are *n* circlets $V_1,V_2,\dots,V_n$. In the star $S_n$, which is defined for $n > 4$, the circlets $U_1,U_2, \dots,U_n$ are put at the intersections of lines $V_{i-2}\,V_{i-1}$ and $V_i\,V_{i+1}$ for $i=1,2,3,\dots,n$ (subscript being taken modulo *n*). In the star $T_n$, $n > 6$, circlets $U_i$ are at the intersections of lines $V_{i-2}\,V_{i-1}$, $V_{i+1}\,V_{i+2}$, for $i=1,2,3,\dots,n$.

An *n*-sided star $S_n$ (or $T_n$) is called a *magic star* and denoted $S_n^M$ (or $T_n^M$) if in its circlets numbers $1,2,3,\dots,2n$ are inscribed in such a way that the sums of the numbers on each line are the same. These stars are sometimes referred to in literature as *David's magic stars*. The sum of the numbers on each line is equal to the double of the sum of all inscribed numbers divided by *n*, i.e. $4n+2$. We call a star $S_n$ (or $T^n$) *weakly-magic* and denote it $S_n^W$ (or $T_n^W$), if we can write into its circlets integers different from each other so that the sum on each line is the same. Each magic star is also weakly-magic, however the reverse implication does not hold.

Let **$M_n$** be a magic square of order *n* and let **$G_{n,n}$** be an complete bipartite graph with vertex-partition $V_1$ {$v_{1,}$ $v_{2,\ \ldots,}$ $v_n$ } and $V_2${$u_1$, $u_2$, …, $u_n$}. If we assign to the edge $v_i$, $u_j$ the number which is in the intersection of *i*-th row and *j*-th column for all *i, j* we obtain a labeling of G by integers so that the sum of labels of every pair of adjacent edges is constant. This fact inspired Czech mathematician Jiří Sedláček in 1962 to introduce the term magic graph.

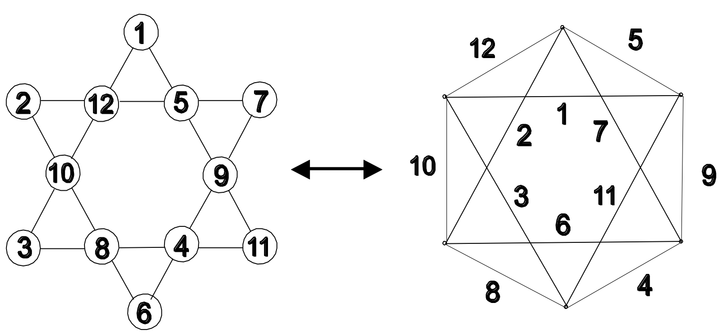
Let **G**(E,V) be an undirected graph without loops and multiple edges. By a magic valuation of **G** we mean a labeling **F** of the edges of **G** by pairwise different positive numbers such that the sum of labels of every pair of adjacent edges is constant. A graph is called *magic* if it allows a magic valuation. Two different characterizations of magic graphs are given in [4] and [5]. A magic graph is called *supermagic* if its edges are labeled by natural numbers 1,2,3,…, |V-1|, |V|.

A spanning subgraph **F** of a graph **G** is called a *(1-2)- factor* of **G** if each of its components is an isolated edge or a circuit. We say that a (1-2)-factor *separates the edges e* and *f* if at least one of them belongs to **F** and neither the edge-part nor the circuit-part contains both of them. In [5] the following theorem is proved.

Theorem 1. A graph **G** is magic if and only if every edge belongs to a (1-2)-factor, and every pair of edges *e, f* is separated by a (1-2)-factor.

Since 1966 many results on magic graphs have been published (see []). The *i*-th power **$G^i$**, i >1, of a graph **G** is the graph with the same vertex set as **G** and such that two vertices of **$G^i$** are adjacent if and only if the distance between these vertices in **G** is at most *i*.

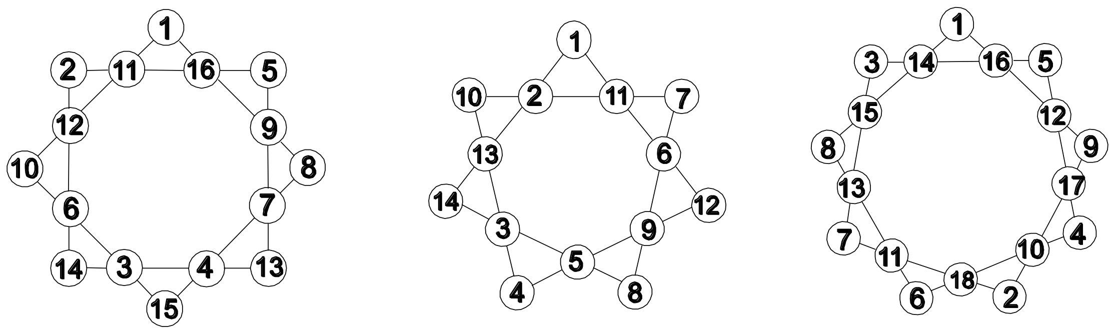
Similarly as magic squares correspond to supermagic complete bipartite graphs, there is a connection between magic stars $S_n$, and a 2nd powers of graph called circles, **$C_n$** (a circle **$C_n$** is a 2-regular graph with *n*-vertices). Into the four circlets on a single line we inscribe the numbers of edges having a certain common vertex.




$S_6^M$ $C_6^2$

Figure 2

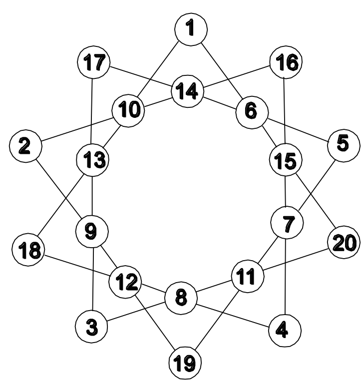
At this time we do not know for which values $n$ the stars $S_n$ are magic. We know that $S_n$ is magic for every $n<27$. In Figure 3 are $S_n^M$ for $n=8,7,9$.



$\mathbb{S}_8^M$ $\mathbb{S}_7^M$ $\mathbb{S}_9^M$

Figure 3

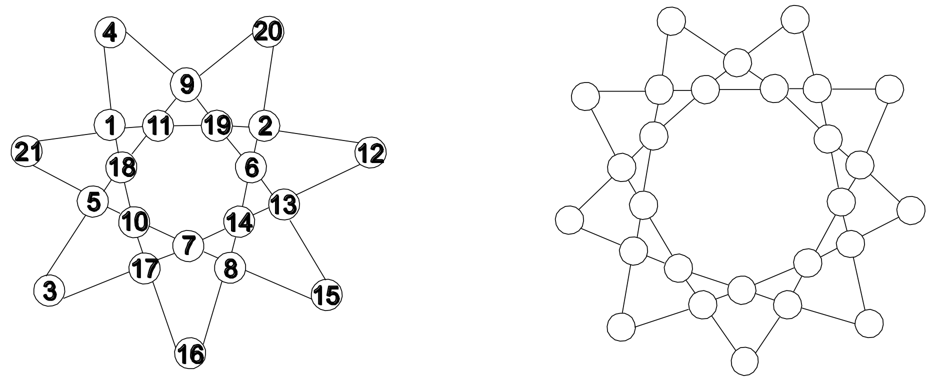
The situation is different for stars $T_n$. These stars correspondent to $\mathbf{C_n}^{\mathbf{3}}$ after omitting all edges joining vertices of distance 2 in $\mathbf{C_n}$ (i.e. the graph $\mathbf{C_n}^{\mathbf{3}} - (\mathbf{C_n}^{\mathbf{2}} - \mathbf{C_n})$). In [6] it is showed that for every even $n$ these stars are magic. We do not know for which odd integers $n$ the $T_n$ is magic. Evidently Theorem 1 implies that $T_n$ are weakly-magic for all $n$.



$\mathbb{T}_{10}^M$

Figure 4

The graph $\mathbf{C_n}^{\mathbf{3}}$ corresponds to the star $ST_n$ showed in Figure 4.




$\mathbb{ST}_7^M$ $\qquad\qquad$ $\mathbb{ST}_9$

Figure 5

From the characterization of magic graphs follows many series of weakly-magic stars of different types. (Many different stars are in [1].)

At the end we add one special magic star, depicted in Figure 7. The sum of the eight numbers on each line is the same. This star corresponds with the graph 4th power of $\mathbf{C_9}$.

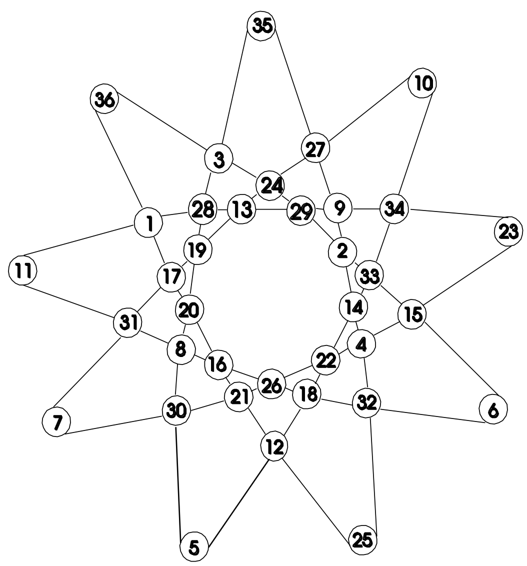



Figure 4

Literature:


1. W.S.Andrews: *Magic squares and Cubes*, Dover 1960
2. H.E.Dudeney: *536 Puzzles and Curious Probléme*, Scribner's 1967
3. J.A.Gallian: *A dynamic survey of graph labeling*, The Electronical Journal of Combinatorics 5(2000), #DS6
4. R.H.Jeurissen: *Magic Graphs, a Characterization*, Europ. J. Combinatorics **9**(1988), 363-368

5. S.Jezný, M.Trenkler: *Characterization of magic graphs,* Czechoslovak Math. Journal **33**(1983), 435-438
6. M.Trenkler: *Magické hviezdy*, Obzory matematiky, fyziky a informatiky **51**(1998), 1-7

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